Gilbert and Porter Chapter 2.3: Rewrite of the proof of the Dual State Lemma

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The proof of the Dual State Lemma in Chapter 2.3 of Gilbert and Porter's *Knots and Surfaces* has some problems with the exposition and a critical typo. This document reworks the proof to make it more correct and comprehensible. (This material starts on page 33.)

Please use the exercises in this document instead of the ones in the book.

Given a state S of a connected diagram D, the dual state $\hat{S}$ is obtained from S by changing *all* the splitting markers.

2.10 Dual State Lemma. *Let S be a state of a connected diagram D and let $\hat{S}$ be the dual state. Suppose that D has r regions. Then $|S| + |\hat{S}| \leq r$.*

We will prove the Dual State Lemma by induction on c, the number of crossings in the diagram D.

First, we establish the basis case:

9. Verify the Dual State Lemma for diagrams with at most two crossings.

We will assume, by way of induction, that if a diagram has $c - 1$ crossings, then the Dual State Lemma holds for that diagram. From this, we wish to prove that the Dual State Lemma holds for the diagram D with c crossings.

Let S be a state of the diagram D. Select one crossing and split D at that crossing both possible ways. Let L be the diagram when you split at that crossing one way, and let R be the diagram when you split at that crossing the other way.

10. Use the connectivity of D to show that at least one of L or R is connected.

Without loss of generality, assume that L is connected.[1] The position of the state marker at which you split D to create L must appear in either S or $\hat{S}$.

[1] Students often get confused by the phrase "without loss of generality". What we mean here is that you can safely call the connected diagram L because the names of the diagrams were arbitrarily assigned. If R ended up being the connected diagram, just go back one step and rename the two diagrams so that L becomes the connected one.

We can assume without loss of generality that it appears in S.[2] After splitting at that one crossing, the remaining state markers in S will be a state of the diagram L which we will call T.

11. Explain why $|T| = |S|$.

12. Show that L has $r - 1$ regions. (Recall that r is the number of regions in the original diagram D.)

13. Why can we assume that $|T| + |\hat{T}| \leq r - 1$?

14. Show that $|\hat{S}| \leq |\hat{T}| + 1$. (Hint: Use Exercise 6 from the book.)

15. You're almost there! Finish off the proof of the Dual State Lemma.

[2] This time, you think about why we can say "without loss of generality".